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DOI:

[10.1007/s10992-016-9395-9](https://doi.org/10.1007/s10992-016-9395-9)

*Document Version*

Peer reviewed version

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*Citation for published version (APA):*

Kürbis, N. (2016). Some Comments on Ian Rumfitt's Bilateralism. *JOURNAL OF PHILOSOPHICAL LOGIC*, 45(6), 623-644. <https://doi.org/10.1007/s10992-016-9395-9>

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# Some Comments on Ian Rumfitt's Bilateralism

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Published in *The Journal of Philosophical Logic* 45/6 (2016): 623–644  
<https://doi.org/10.1007/s10992-016-9395-9>

## Abstract

Ian Rumfitt has proposed systems of bilateral logic for primitive speech acts of assertion and denial, with the purpose of ‘exploring the possibility of specifying the classically intended senses for the connectives in terms of their deductive *use*’ (Rumfitt (2000): 810f). Rumfitt formalises two systems of bilateral logic and gives two arguments for their classical nature. I assess both arguments and conclude that only one system satisfies the meaning-theoretical requirements Rumfitt imposes in his arguments. I then formalise an intuitionist system of bilateral logic which also meets those requirements. Thus Rumfitt cannot claim that only classical bilateral rules of inference succeed in imparting a coherent sense onto the connectives. My system can be extended to classical logic by adding the intuitionistically unacceptable half of a structural rule Rumfitt uses to codify the relation between assertion and denial. Thus there is a clear sense in which, in the bilateral framework, the difference between classicism and intuitionism is not one of the rules of inference governing negation, but rather one of the relation between assertion and denial.

## 1 Introduction

Ian Rumfitt has argued that specifying the meanings of the logical constants in terms of primitive speech acts assertion and denial meets a well known Dummettian challenge: to provide a semantic theory that specifies their meanings in terms of their use in deductive arguments that justifies classical logic, while at the same time satisfying the requirement of harmony between introduction and elimination rules. Rumfitt proposes a formal system of *bilateral* logic that captures his fundamental idea in containing rules which specify not only the grounds on which formulas can be asserted and the consequences of asserting them, but also grounds and consequences of denying formulas. Rumfitt’s logic is classical and there is harmony between introduction and elimination rules. Thus Rumfitt’s framework justifies classical logic.

Dummett’s challenge can be understood in a stronger and in a weaker sense. The weaker one is only to provide a semantic theory satisfying Dummettian requirements that justifies classical logic, where it is left open whether intuitionist logic can equally be justified—we are then free to adopt classical logic on

methodological grounds, or maybe we can adopt different logics for different purposes. The stronger one is to provide a semantic theory on which classical logic is justified and intuitionist logic shown to be defective. Correspondingly, Rumfitt's objective can also be understood in two ways. He sometimes seems to present himself as aiming to meet the weaker challenge, sometimes the stronger one. On the one hand, Rumfitt initially introduces his bilateral conception of sense merely as a 'framework on which it is possible to specify "directly" the sense of a classical negation sign' (Rumfitt (2000): 797), without raising the question of excluding a similar intuitionist conception. On the other hand, as Rumfitt's account evolves, a stronger claim is put forward, namely that 'it can be shown that bilateral rules will not endow " $\neg$ " with a coherent sense unless double negation elimination is valid' (Rumfitt (2000): 814), which emulates some of Dummett's claims to the effect that classical negation is strictly speaking unintelligible.

In this paper, I'll focus on the stronger challenge. It is, I think, the philosophically more interesting one and more in line with what I take to be Dummett's own intentions, whose justification of intuitionist logic at the same time shows classical logic to be defective, as on his account, the rules for classical negation are not harmonious. It is also the reading I would argue is in the foreground of Rumfitt's discussion. He acknowledges that intuitionist logic may be justified, but his remarks on that issue make it clear that he thinks that this is relative only to a unilateral framework. Thus, I take Rumfitt's position to be that bilateralism justifies classical logic, but intuitionist logic is defective in this framework, while in a unilateral framework, it is the other way round.<sup>1</sup> Which logic to apply to an area of discourse then depends on whether it is best represented in the bilateral or in the unilateral framework, i.e. depending on whether assertion and denial are equally fundamental in a given area of discourse or not. Rumfitt discusses discourse about the past as an area that should be formalised bilaterally, whereas he speculates that mathematics may be a case that should be treated unilaterally (Rumfitt (2000): 817ff). In this paper, I will argue, against this understanding of Rumfitt's objective, that it is possible to formalise an intuitionist bilateral logic in which the logical constants are governed by harmonious rules of inference. Thus the bilateral framework does not exclude intuitionist logic and Rumfitt cannot claim to have met the strong reading of Dummett's challenge.

## 2 Classical Bilateralism

### 2.1 Rumfitt's Two Formalisations of Bilateral Logic and Two Arguments for Classicality

Rumfitt gives two formalisations of bilateral logic and two arguments for their classical nature in (Rumfitt (2000)). To get clear about what a bilateral formalisation of intuitionist logic has to achieve in order to have the same meaning-theoretical status as Rumfitt's classical systems, I'll begin by considering which of Rumfitt's formalisations is the pertinent one, given Rumfitt's meaning-theoretical concerns. The system of intuitionist bilateral logic to be proposed at the end of this paper will emulate its form. My evaluation of Rumfitt's

<sup>1</sup>Cf. (Rumfitt (2000): 805f), from where I quote later in this paper.

formalisations is based on his two arguments for their classical nature: in fact, only one of the formalisations meets the requirements on a meaning-theoretically satisfactory bilateral system imposed in these arguments.

### 2.1.1 Two Formalisations of Bilateral Logic

The rules of Rumfitt's bilateral logics apply to *signed formulas*, indicating one of two speech acts. Asserted formulas are signed by +, denied ones by -. Lower case Greek letters range over signed formulas.  $\alpha^*$  designates the *conjugate* of  $\alpha$ , the result of reversing its sign from + to - and conversely. Rumfitt's second formalisation,  $\mathfrak{J}$ , essentially the system proposed in (Smiley (1996): 5), is straightforward:

$+\&I$ :	$+A, +B \vdash +A\&B$	
$+\&E$ :	$+A\&B \vdash +A$	$+A\&B \vdash +B$
$-\vee I$ :	$-A, -B \vdash -A \vee B$	
$-\vee E$ :	$-A \vee B \vdash -A$	$-A \vee B \vdash -B$
$+\supset E$ :	$+A \supset B, +A \vdash +B$	
$-\supset E$ :	$-A \supset B \vdash +A$	$-A \supset B \vdash -B$
$+\neg I$ :	$-A \vdash +\neg A$	
$+\neg E$ :	$+\neg A \vdash -A$	
Reductio:	If $\Gamma, \alpha \vdash \perp$ , then $\Gamma \vdash \alpha^*$	
Non-Contradiction:	From $\alpha, \alpha^*$ , infer $\perp$	
Structural Rules:	Thinning, Cut, Reflexivity	

Rumfitt calls Reductio and Non-Contradiction *co-ordination principles* (Rumfitt (2000): 804).

Rumfitt's first formalisation contains, for each connective, two sets of rules that specify the grounds for asserting/denying complex formulas and the consequences of asserting/denying them (Rumfitt (2000): 800ff). I call this set of rules  $\mathfrak{I}\mathfrak{D}$ :

$+\&I$ :	$\frac{+A \quad +B}{+A\&B}$	$+\&E$ :	$\frac{+A\&B}{+A} \quad \frac{+A\&B}{+B}$
$-\&I$ :	$\frac{-A}{-A\&B} \quad \frac{-B}{-A\&B}$	$-\&E$ :	$\frac{-A\&B \quad \frac{\overline{-A}^i \quad \Pi}{\phi} \quad \frac{\overline{-B}^i \quad \Sigma}{\phi}}{\phi}^i$
$+\vee I$ :	$\frac{+A}{+A \vee B} \quad \frac{+B}{+A \vee B}$	$+\vee E$ :	$\frac{+A \vee B \quad \frac{\overline{+A}^i \quad \Pi}{\phi} \quad \frac{\overline{+B}^i \quad \Sigma}{\phi}}{\phi}^i$
$-\vee I$ :	$\frac{-A \quad -B}{-A \vee B}$	$-\vee E$ :	$\frac{-A \vee B}{-A} \quad \frac{-A \vee B}{-B}$

$$\begin{array}{ll}
+ \supset I: \frac{\frac{\overline{+A}^i}{\Pi} \quad +B}{+A \supset B}^i & + \supset E: \frac{+A \supset B \quad +A}{+B} \\
- \supset I: \frac{+A \quad -B}{-A \supset B} & - \supset E: \frac{-A \supset B \quad -A \supset B}{+A \quad -B} \\
+ \neg I: \frac{-A}{+ \neg A} & + \neg E: \frac{+ \neg A}{-A} \\
- \neg I: \frac{+A}{- \neg A} & - \neg E: \frac{- \neg A}{+A}
\end{array}$$

Gibbard points out that these rules alone do not yield classical logic, but a constructive logic with strong negation in which, although double negation elimination and DeMorgan's Laws hold, the laws of excluded middle and non-contradiction do not (Gibbard (2002): footnote 2). Rumfitt responds that he intended  $\mathfrak{A}\mathfrak{D}$  to be 'animated by the rule [he] called Smileyian Reductio' (Rumfitt (2002): footnote 2), and which I here call simply Smiley to avoid confusion with Reductio:

Smiley: If  $\Gamma, \alpha \vdash \beta$  and  $\Gamma, \alpha \vdash \beta^*$ , then  $\Gamma \vdash \alpha^*$

This is surprising. As I will discuss in the next section, one of Rumfitt's arguments for the classical nature of bilateral logic is based on a proof, which appeals to  $-\neg E$ , that it satisfies the requirement that Non-Contradiction can be restricted to atomic signed formulas (Rumfitt (2000): 815f). Thus, we should expect that Non-Contradiction and Reductio, rather than Smiley, are added to  $\mathfrak{A}\mathfrak{D}$  to formalise classical bilateral logic. Rumfitt does not establish the corresponding claim that Smiley can be restricted to atomic premises  $\beta$ . It is, however, easily seen to hold (see Appendix 5.1).

A more significant reason why Rumfitt's choice of Smiley over Non-Contradiction and Reductio is surprising is that he devotes some discussion to the status of  $\perp$ , which does not occur at all in the formalisation that uses Smiley. Rumfitt follows Neil Tennant's proposal (Tennant (1999)) and sees  $\perp$  not as a proposition or a speech act, but as a 'punctuation mark in the deduction' indicating a 'dead end' (Rumfitt (2000): 794).

Ferreira observes that Non-Contradiction is superfluous, given Rumfitt's understanding of  $\perp$ : 'it is merely a punctuation sign indicating that we have reached deductions of  $\alpha$  and  $\alpha^*$ , for some signed atomic sentence  $\alpha'$ ' (Ferreira (2008): 1056). Instead of signalling that we reached a certain stage, we can leave it at having reached it and proceed immediately to the next step. We can then show that the same point has been reached when complex conjugates have been derived. There is some justice to Ferreira's observation, but the form of Reductio means that Non-Contradiction is not entirely superfluous: it is needed to reach conclusions of the form  $\perp$ , which are required to apply Reductio. Without Non-Contradiction, we would expect, rather than Reductio, a rule that specifies directly how to proceed when we have reached a dead end. Smiley does exactly that. Smiley is a consequence of Non-Contradiction and Reductio, and they can

be replaced with Smiley. Thus the use of  $\perp$  in the formal system is superfluous, and questions concerning the status of  $\perp$  are irrelevant. This is just as well. Nothing in the rules determines that  $\perp$  is a punctuation mark, i.e. very different from the speech acts that are the premises and conclusions of the other rules: it performs the same role as the assertion of a contradiction or the denial of a tautology. Considerations concerning the status of  $\perp$  do not give reasons to favour one of Smiley and Non-Contradiction+Reductio over the other.

As Smiley is a consequence of Non-Contradiction and Reductio, Rumfitt could also have said that these two rules animate  $\mathfrak{A}\mathfrak{D}$ . At the same time, the importance of restricting Non-Contradiction to atomic premises should carry over to Smiley, which can also be restricted to atomic premises, given  $\mathfrak{A}\mathfrak{D}$ . From this perspective, it does not matter whether we add Smiley or Non-Contradiction and Reductio to  $\mathfrak{A}\mathfrak{D}$ , and there are two formally and philosophically equivalent ways of formalising Rumfitt's second system, which I call  $\mathfrak{B}$ :  $\mathfrak{A}\mathfrak{D} + \text{Smiley}$  and  $\mathfrak{A}\mathfrak{D} + \text{Non-Contradiction} + \text{Reductio}$ .  $\mathfrak{B}$  and  $\mathfrak{I}$  capture the same consequence relation.

### 2.1.2 Two Arguments for Classicality

In this section, I'll discuss the two arguments Rumfitt gives to justify classical logic in his bilateral framework. After that, I'll decide on the basis of these arguments which of Rumfitt's two formalisations of bilateral logic is the primary one,  $\mathfrak{I}$  or  $\mathfrak{B}$ . The arguments also indicate requirements an intuitionist bilateral system has to fulfil. For ease of reference, I'll label them ARGUMENT 1 and ARGUMENT 2. In each case, I'll begin by stating the essence of the argument in a few sentences. I then move on to give the details, quoting extensively from Rumfitt.

Rumfitt's first argument has no direct parallel in Dummett's work. It is based on a restriction Rumfitt imposes on the rule of Non-Contradiction:

ARGUMENT 1. Bilateral rules for connectives succeed in specifying their senses coherently only if it can be shown that Non-Contradiction holds for complex formulas, if it holds for atomic ones. The proof that this is so appeals to the characteristically classical inference  $\neg A \vdash + A$ , which entails double negation elimination. Hence bilateral logic is classical.

The details of the argument are contained in the following passage:

'within a bilateral framework it is indeed natural to [...] take the rules  $+ \neg I$  and  $+ \neg E$  to specify the sense of " $\neg$ ". (This is why [Rumfitt] took these rules to be the primitive rules for negation in the system  $\mathfrak{I}$ .) This of course ensures that  $+ (\neg A) \dashv \vdash - A$ . The question we have to address is why such a specification of sense for " $\neg$ " should require that  $- (\neg A) \dashv \vdash + A$ . [...] An adequate specification of the sense of any negation operator  $\mathbf{O}$  ought to entail, for any sentence  $A$ , that  $A$  and  $\mathbf{O}A$  are contradictory. [...] However, the principle of consistency is not similarly a component of the bilateral specification of the sense of " $\neg$ " [as it is for intuitionist negation

in the unilateral framework]. We have therefore to consider how it may be derived. [...] The conditions for affirming and rejecting each atomic sentence  $\lceil P \rceil$ —although *ex hypothesi* independent—still need to be co-ordinated by [Non-Contradiction]. [...] The question arises whether complex formulae are similarly co-ordinated. Given that  $+ (\neg A)$  and  $- A$  are interdeducible, such co-ordination will be necessary if  $+ A$  and  $+ (\neg A)$  are themselves to be contradictory. Accordingly, if the principle of consistency is to be derived from the specifications of sense for the connectives, as we want it to be, we shall need to be able to show that those specifications entail the co-ordination principle  $+ A, - A \vdash \perp$  for each well-formed formula  $A$ , given the information that all the atoms are co-ordinated. It is in showing this that the rule  $\neg\neg E$  is needed.’ (Rumfitt (2000): 814ff)

In other words, according to Rumfitt, he is on a par with the intuitionist in that they both need to stipulate that the atomic sentences are coordinated by their respective versions of Non-Contradiction. However, the intuitionist builds Non-Contradiction for all sentences into his account of negation, whereas Rumfitt needs to show that it holds for complex ones, if it holds for the atomic ones. To show this, Rumfitt appeals to  $\neg\neg E$ . This is the negation rule that is at issue between classicists and intuitionists. Rumfitt concludes, ‘the coherence of a bilateral model of sense requires  $\neg\neg E$ —and hence requires double negation elimination’ (Rumfitt (2000): 817). Within a bilateral system, ‘the distinctively classical method of eliminating “ $\neg$ ” [i.e. double negation elimination] is a consequence of a rule that is needed to ensure that the co-ordination principle (and hence the principle of consistency) holds for complex formulae as well as for atomic ones’ (Rumfitt (2000): 816). ‘This observation, [Rumfitt suggests], provides some kind of justification for the rule of double negation elimination—a justification which makes no appeal to the law of bivalence.’ (Rumfitt (2002): 309) Thus bilateral logic is classical.

Rumfitt’s second argument makes use of Dummett’s requirement that harmony must obtain between introduction and elimination rules for logical constants, if they are to impose a coherent meaning on them:

ARGUMENT 2. The rules for classical negation in a bilateral system are in harmony, which is what Dummett requires if the sense of a connective is to be specified completely by rules of inference governing it. Bilateral rules for intuitionist negation are not in harmony. So the bilateral framework justifies classical logic, but not intuitionist logic.

The details are the following. Rumfitt agrees with Dummett that in a unilateral framework, ‘the classical deduction rules [...] exhibit a peculiar anomaly—an anomaly which is located in the classical elimination rule for negation’ (Rumfitt (2000): 805). For rules governing a connective to specify its meaning, the grounds of inferring a formula with that connective as main operator and the consequences of inferring it must be in harmony. In a unilateral system, the rules for a connective specify under which conditions a formula with that connective as main operator may be asserted and the consequences of asserting such a

formula. Correspondingly, in a bilateral system, the rules specify the conditions under which such a formula may be asserted and denied and what follows from its assertion and denial. The rules for classical logic in a unilateral system are not in harmony. By contrast,

‘in a bilateral system, the situation is reversed: it is the intuitionistic rules for negation that are anomalous. The four bilateral rules given for “ $\neg$ ” [in  $\mathfrak{B}$ ] comprise: a rule that entitles a reasoner to affirm a negated formula; a rule that says what he is entitled to do having made such an affirmation; and cognate rules dealing with the conditions for and the consequences of rejecting a negated formula. These four rules exhibit a kind of harmony and [ARGUMENT 1 shows that] there is a clear sense in which they are all needed to endow a negation sign with a coherent bilateral meaning.’ (Rumfitt (2000): 806)

Rumfitt does not formalise an intuitionist bilateral logic,<sup>2</sup> but it would lack  $\neg\neg E$ , while the other three negation rules are intuitionistically acceptable. These three rules don’t exhibit the same symmetry as the four rules.

The passage just quoted also provides a rationale for my focus on the strong reading of Dummett’s challenge and that it is this one that Rumfitt aims to address. Rumfitt repeats the point later.

‘The disharmony which Dummett discerns between the classical rules for negation when these are formalized in the standard, purely affirmative, presentation of a logical system is now beside the point. [...] It is hard to see how any pair of introduction and elimination rules could be more harmonious, or more stable, than [the four negation rules of  $\mathfrak{B}$ ].’ (Rumfitt (2002): 308)

‘If we formalize logic in the bilateral style, then it is the classical operational rules that exhibit the analogue of harmony, while the intuitionist rules for negation do not.’ (Rumfitt (2008): 1062)

Thus, following Dummett, if to specify the meanings of the logical constants purely in terms of rules of inference requires these to be in harmony, only classical negation has a coherent meaning in a bilateral system, but not intuitionist negation.

With this material in place, let’s have a closer look at Rumfitt’s two formalisations to determine which is the meaning-theoretically fundamental one. After this, I’ll assess whether the two arguments are equally convincing.

## 2.2 $\mathfrak{I}$ vs. $\mathfrak{B}$

Although  $\mathfrak{I}$  and  $\mathfrak{B}$  capture the same consequence relation, they cannot share the same meaning-theoretical status. Rumfitt shares Dummett’s view that the

<sup>2</sup>His claim that an intuitionist bilateral logic results from  $\mathfrak{I}$  by restricting Reductio to ‘If  $\Gamma, + A \vdash \perp$ , then  $\Gamma \vdash - A$ ’ (Rumfitt (2000): 806) is a slip of the pen. We should have  $+ \neg\neg A \vdash + \neg(A \supset \neg A)$  in intuitionist  $\mathfrak{I}$ . By  $+ \neg E$  and Cut  $+ \neg\neg A \vdash - (A \supset \neg A)$ , and by  $- \supset E$  and Cut  $+ \neg\neg A \vdash + A$ . Intuitionist  $\mathfrak{I}$  cannot have  $- \supset E$  and also needs rules to derive asserted disjunctions.



meanings of logical constants are given by rules of inference. But which rules? Those of  $\mathfrak{I}$  or those of  $\mathfrak{B}$ ? Sometimes Rumfitt claims that the four rules of  $\mathfrak{ID}$  governing a connective are necessary to specify its bilateral meaning. At other times he claims that the two rules of  $\mathfrak{I}$  are sufficient. It is not possible for both claims to be correct. Rumfitt needs to make a decision which system it is that specifies the meanings of connectives bilaterally.

Notice that  $\mathfrak{I}$  and  $\mathfrak{B}$  cannot be equally successful in specifying the meanings of connectives in the way that different axiomatisations can capture the same semantic structure. For that, a semantic structure to be captured has to be given independently. Rumfitt has not given such a structure. Doing so would be counterproductive: his aim is to specify the meanings of the connectives in a way that ‘gives a connective’s sense—not by directly assigning any semantical value to it—but rather by characterising a putatively basic aspect of its deductive use.’ (Rumfitt (2000): 788)

$\mathfrak{I}$  has some nice features. Its rules are natural and allow the normalisation of deductions in the following sense. Let a maximal formula of a deduction in  $\mathfrak{I}$  be one that is the major premise of an elimination rule for its main connective and either the conclusion of an introduction rule for the main connective or the conclusion of Reductio. Any maximal formula can be removed from deductions in  $\mathfrak{I}$  (see Appendix 5.2). Thus there is a sense in which introduction rules of  $\mathfrak{I}$  are balanced by the elimination rules and Reductio.

Deductions in  $\mathfrak{B}$  normalise, too, however, so the fact that they normalise in  $\mathfrak{I}$  cannot help deciding between  $\mathfrak{I}$  and  $\mathfrak{B}$ . The sense in which rules of  $\mathfrak{I}$  are balanced is weaker than the sense Dummett requires, namely that the rules are in harmony, which entails normalisability and is exhibited by the rules of  $\mathfrak{B}$ . In the light of ARGUMENT 2, we should expect that it is  $\mathfrak{B}$  which specifies the meanings of the connectives bilaterally. Rumfitt describes  $\mathfrak{ID}$  as ‘unwieldy’.  $\mathfrak{I}$  is presented as a more compact ‘sound and complete axiomatisation of the classical propositional calculus. [However, Rumfitt claims] that it provides much more than that—viz., a direct specification of the senses of the connectives in terms of their deductive use.’ (Rumfitt (2000): 805) The focus on  $\mathfrak{I}$  during the rest of the paper shows that this is not merely a slip of the pen. This is puzzling for a number of reasons.

If  $\mathfrak{I}$  achieves the goal of determining the meaning of negation in a bilateral framework, then what speaks in favour of a classical bilateral logic cannot be that its negation rules exhibit a kind of harmony that an intuitionistic version would not exhibit. Even though  $\neg\neg E$ —the rule around which a justification of classicality must centre—stands in a certain harmony with  $\neg I$ , where the former is intuitionistically unacceptable and the latter uncontroversial, this harmony cannot be the reason for adopting  $\neg\neg E$ , because  $\neg I$  is not a rule of  $\mathfrak{I}$ . If  $\mathfrak{I}$  succeeds in specifying the meaning of  $\neg$ , then the harmony exhibited by the four rules of  $\mathfrak{ID}$  cannot play a role in justifying the classical behaviour of a negation whose meaning is specified bilaterally. Furthermore, although the rules for  $\&$ ,  $\vee$  and  $\neg$  of  $\mathfrak{I}$  can be said to be in harmony, it has only elimination rules for  $\supset$ , so there is no harmony, as this is a relation between introduction and elimination rules.

$\mathfrak{I}$  does not contain primitive rules for all connectives to specify the grounds and consequences for assertions *and* denials for formulas. Where they are lacking, such rules can of course be derived. To illustrate, consider  $\neg\neg E$ : by  $\neg I$ ,  $\neg A \vdash \neg\neg A$ . Assume  $\neg\neg A$ . By Non-Contradiction,  $\perp$ , so by Reductio  $A$ ,

discharging  $\neg A$ . However, this derivation illustrates that Non-Contradiction cannot be reduced to atomic premises in  $\mathfrak{S}$ . Having arrived at a denied negated formula, the only rules of  $\mathfrak{S}$  other than Non-Contradiction that can be applied are  $+\&I$ ,  $-\vee I$  or  $+\neg I$ , none of which decompose the premises. Hence we cannot establish the induction step of a proof that Non-Contradiction can be restricted to atomics, as that requires decomposition of premises into their constituents. In fact, in the context of  $\mathfrak{S}$ , Non-Contradiction does the job of an elimination rule for denied negations and conjunctions and for asserted disjunctions and implications. Consequently, Rumfitt's ARGUMENT 1 for the classical nature of bilateral systems—that bilateral logic has to be classical as  $\neg\neg E$  is needed to show that Non-Contradiction can be restricted to atomic premises—cannot get off the ground if  $\mathfrak{S}$  was the pertinent formalisation, as in this system Non-Contradiction cannot be restricted to atomic premises.

There is another reason against  $\mathfrak{S}$  as the system that specifies the bilateral meanings of the connectives. Its rules only exhibit the harmony that the rules of  $\mathfrak{B}$  exhibit if we take into account derived rules of inference. However, derived rules of inference cannot play a role in specifying the sense of a connective. Consider adding  $\Gamma, \neg\neg\neg A \vdash \neg A$  and  $\Gamma, \neg A \vdash \neg\neg\neg A$  to intuitionist logic. Although in harmony, clearly these rules won't add anything to the meaning of negation as already specified by its usual introduction and elimination rules. Derived rules of inference cannot determine the meanings of connectives. They hold *in virtue of* the meaning of connectives, as determined by other rules. Thus Rumfitt cannot stipulate that the four rules of  $\mathfrak{B}$  need not all be primitive rules of the system that specifies the bilateral meanings of connectives, but that some of them may be derived rules.

It is plausible in itself, given Rumfitt's meaning-theoretic concerns, that the rules that specify the sense of connectives must be primitive harmonious rules, rather than partially derived ones. It is even more salient in the particular case of bilateralism. If harmonious rule were allowed to be partially derived, rather than primitive, it would follow that it cannot be necessary either to specify primitively what follows from a rejected complex sentence in general; it should always be sufficient if these rules may be derived. Given the co-ordination principles and the *assertive* rules for the constants, all the *rejective* rules may be derived. The distinctively bilateralist thesis could not add anything to the way the meanings of connectives are to be determined. We might as well follow Frege and define denial in terms of assertion and negation.

Thus neither ARGUMENT 1 nor ARGUMENT 2 go through for  $\mathfrak{S}$ . Hence, contrary to what Rumfitt claims,  $\mathfrak{S}$  is not a good candidate for a system that specifies the meanings of the connectives bilaterally. We must conclude that  $\mathfrak{S}$  merely provides convenient shortcuts for carrying out deductions, while the correct way of specifying the bilateral senses of the connectives is given by  $\mathfrak{B}$ .

The only thing that speaks against  $\mathfrak{B}$  is its unwieldiness. In the presence of Smiley or Non-Contradiction+Reductio, the rules of  $\mathfrak{A}\mathfrak{D}$  carry a high amount of redundancy: a more economic choice of primitive rules is possible, where half the rules of  $\mathfrak{A}\mathfrak{D}$  are derivable, even while restricting Smiley/Reductio to atomic premises (i.e. the formalisation  $\mathfrak{R}$  given at the end of this section). Maybe a case can be made that therefore it does not capture our practice adequately. Arguably, we do not have primitive practices governing the use of each connective in assertions and denials, although we can readily comprehend each such case on the basis of more primitive examples, i.e. as practices derived from more

primitive practices. Rumfitt demands that the rules of a system that specifies the meanings of the connectives are primitively obvious in Peacocke's sense: 'a connective will possess the sense that it has by virtue of its competent users' finding certain rules of inference involving it to be primitively obvious.' (Rumfitt (2000): 787) Maybe our practice favours assertive rules for conjunctions. The evidence may be less conclusive for disjunction, but maybe competent users of  $\vee$  need not find the assertive elimination rule, proof by cases, primitively obvious. The assertive introduction rule may have a similar status, as it allows to disjoin arbitrary sentences to a given sentence. Maybe this is a remote aspect of our practice. The rejective rules of  $\exists$ , however, may be something any competent user of  $\vee$  accepts immediately. For negation, we may simply have a choice of which of the four rules to use or we may use them all, even though that results in some redundancy. The problem with this line of argument lies in the implication rules of  $\exists$ . It is questionable whether competent users of implication find the  $- \supset E$  rules of  $\exists$  primitively compelling. Arguably, our practice with conditionals favours *modus ponens* and *modus tollens*, but as these rules are equivalent in  $\exists$ ,  $- \supset E$  cannot be replaced by a bilateral version of *modus tollens*.

Be that as it may, the two previous points remain. Even if a case for  $\exists$  could be made on the basis that it represents the primitives of our inferential practice more adequately than  $\mathfrak{B}$ , what would then speak in favour of  $\exists$  is given much less weight by Rumfitt than the requirement of harmony and his proof that Non-Contradiction can be restricted to atomics.

Should Rumfitt insist on  $\exists$  as the system that specifies the bilateral meanings of the connectives, I propose that an alternative system fares just as well. The proof that in  $\mathfrak{B}$ , Non-Contradiction/Smiley can be restricted to atomic premises uses only assertive and rejective elimination rules for the connectives. Let  $\mathfrak{N}$  be the system  $+ \neg E, - \neg E, + \& E, - \& E, + \vee E, - \vee E, + \supset E, - \supset E$ , Cut, Thinning, Reflexivity, and either Reductio+Non-Contradiction or Smiley (the latter two restricted to atomic premises).  $\mathfrak{N}$  is deductively equivalent to  $\exists$ . ARGUMENT 1 goes through for it. Deductions normalise in the sense that formulas which are conclusions of Reductio/Smiley and major premises of an elimination rule can be removed. The only disadvantage it has over  $\exists$  is that maybe it does not represent our practice quite as neatly as  $\exists$  in terms of rules that competent users may find primitively compelling, due to  $- \& E$ . However,  $\mathfrak{N}$  also has advantages over  $\exists$ . It has assertive and rejective rules for each connective. It thereby shows directly how each connective is embedded into the practices of asserting and denying. Thus bilateralism may go well with the view that the elimination rules determine the meanings of the logical constants.

### 2.3 Argument 1 vs. Argument 2

Having established that  $\mathfrak{B}$ , not  $\exists$ , is the system that specifies the bilateral meanings of the connectives, I'll now go on to assess Rumfitt's two arguments for classicality. The importance Rumfitt assigns to the proof showing that Non-Contradiction can be restricted to atomic formulas was that  $- \neg E$  is used in the induction step. He concludes that the rule is required for a coherent specification of the meanings of negation in a bilateral framework. This has been challenged by Gibbard, as a weaker rule suffices for the induction step to go through (Gibbard (2002): 299):

Gibbard's  $\neg\neg E$ : From  $\neg A$  and  $\neg\neg A$  infer  $\perp$

Against Gibbard's alternative, Rumfitt repeats that 'an adequate specification of bilateral senses for the sentences of a language must ensure that each sentence possess such a sense, and this will involve associating with each sentence determinate conditions under which it may correctly be asserted and determinate conditions under which it may correctly be denied.' (Rumfitt (2002): 310) Gibbard's alternative does not fulfil this requirement, as his rules 'fail to specify the conditions for denying a negated sentence.' (Rumfitt (2002): 311) Thus they fail to specify the bilateral meaning of  $\neg$  adequately.

Rumfitt's argument, however, does not exclude that there are yet other  $\neg\neg E$  rules that fulfil the requirement of specifying the condition for denying negated sentences and allow Non-Contradiction to be restricted to atomics. All we have been given is the negative point that Gibbard's rule does not suffice to specify the meaning of  $\neg$  in a bilateral framework. What is lacking is a positive case for  $\neg\neg E$ , which is what Rumfitt needs to establish the claim that 'an adequate specification of the bilateral sense of [the negation] sign will validate double negation elimination' (Rumfitt (2002): 311).

Rumfitt does, of course, provide the material for a positive argument for  $\neg\neg E$ . In his reply to Gibbard, Rumfitt recalls that 'the rationale [for the bilateral account] was [...] to articulate a notion of sense which meets Dummett's *desideratum* of being closely connected to the way a sentence is actually used, but within which the behaviour of the classical negation operator would not be suspect or anomalous.' (Rumfitt (2002): 309) Whatever the rule, it needs to balance  $\neg\neg I$  in such a way that the grounds for denying  $\neg A$  are in harmony with the consequences of denying it. This is achieved by Rumfitt's  $\neg\neg E$ , but not Gibbard's.

Without an appeal to harmony, Rumfitt would need to give a different explanation of what the anomaly or suspect behaviour of an operator might consist in. Rumfitt offers some such explanation when he complains that 'in a bilateral framework, one will wish to know why the intuitionistic logician has no general account to offer of the consequences of rejecting a negated sentence or formula.' (Rumfitt (2000): 806) But this by itself hardly brings out an anomaly in a potential intuitionistic bilateral logic. The challenge is not too hard to answer. Certainly the bilateral intuitionist may have a general account: one option is that if you reject the negation of  $A$ , you may assert its double negation. This is as general an account of what follows from a rejected negated formula as one can wish for. The anomaly Rumfitt mentions arises only in relation to the requirement of harmony between introduction and elimination rules, which demands the general account to come as a rule of a certain form. Given  $\neg\neg I$ , the grounds for denying a negated sentence are balanced by the consequences of denying it as specified by  $\neg\neg E$ , which are therefore in harmony.

The requirement of harmony provides Rumfitt with a direct argument for  $\neg\neg E$ . This means, however, that to back up ARGUMENT 1, Rumfitt needs to appeal to ARGUMENT 2. ARGUMENT 1, then, which aimed to establish that  $\neg\neg E$  is needed to endow negation with a coherent bilateral sense, is of no importance, as to exclude other rules for rejected negations like Gibbard's, Rumfitt needs to move to ARGUMENT 2. Thus the requirement of harmony is the only argument Rumfitt gives in favour of  $\neg\neg E$ .

There is, however, a drawback for Rumfitt. Harmony establishes that  $\neg\neg E$

is the correct rule to balance  $\neg\neg I$ , but this doesn't establish that it is the only harmonious pair of rules for rejected negations. We can interpret Rumfitt as posing the intuitionist a challenge, just as Dummett has posed the classicist a challenge: to provide harmonious bilateral rules that yield intuitionist, rather than classical, logic. To meet this challenge, the intuitionist has the option of modifying the introduction rule for rejected negations so that it is harmonious with an intuitionistically acceptable elimination rule for rejected negations, rather than  $\neg\neg E$ . If it is also possible to formulate alternative pairs of harmonious rules for rejected conjunctions and disjunctions that are intuitionistically acceptable, then the intuitionist will have shown that it is possible to give the intuitionist connectives a coherent bilateral sense. I will formalise precisely such a system in the next section, following the format of  $\mathfrak{B}$ , which I argued is the meaning-theoretically pertinent one of Rumfitt's two formalisations.

### 3 Intuitionist Bilateralism

To show that ARGUMENT 2 does not establish that bilateral logic must be classical, it suffices to formalise a bilateral intuitionist logic where all logical constants are governed by harmonious rules. Humberstone formalises a system of bilateral intuitionist logic (Humberstone (2000): 364f), which uses the assertive rules for  $\&$ ,  $\vee$  and  $\supset$ ,  $\neg\neg I$ ,  $\neg\neg E$ , Cut, Transitivity, Reflexivity and the following two rules

*Ex contradictione quodlibet*: From  $\alpha, \alpha^*$  infer  $\beta$

Intuitionist Smiley: From  $\Gamma, +A \vdash \alpha$  and  $\Gamma, +A \vdash \alpha^*$  infer  $\Gamma \vdash -A$

Instead of these, we can also use corresponding rules with  $\perp$ , i.e. Non-Contradiction, the intuitionist half of Reductio 'From  $\Gamma, +A \vdash \perp$ , infer  $\Gamma \vdash -A$ ' and *ex falso quodlibet* 'From  $\perp$  infer  $\beta$ '.

The rules of Humberstone's system, like those of  $\mathfrak{I}$ , do not specify primitively the grounds and consequences of asserting as well as denying formulas. The challenge is to produce an intuitionist version of  $\mathfrak{ID}$  with harmonious rules. My intuitionist bilateral system  $\mathfrak{Int}\text{-}\mathfrak{B}$  results from Humberstone's system by adding the rejective disjunction rules of  $\mathfrak{ID}$  and the following rejective rules for  $\&$ ,  $\supset$  and  $\neg$ :

$$\begin{array}{c}
\frac{\frac{\overline{+A}^i}{\Pi} \quad \frac{-B}{-A\&B}^i}{-A\&I:} \\
\frac{\frac{\overline{-A}^i}{\Pi} \quad \frac{\overline{-A}^i}{\Pi} \quad \frac{-B}{\alpha} \quad \frac{\alpha^*}{\alpha^*}^i}{-A\supset I:} \\
\frac{\frac{-A\supset B}{\beta} \quad -A}{-A\supset E:} \quad \frac{-A\supset B}{-B}
\end{array}$$

$$\begin{array}{c}
\frac{\frac{\overline{-A}^i}{\Pi} \quad \frac{\overline{-A}^i}{\Pi}}{\frac{\alpha}{\alpha^*}^i} \quad -\neg I: \quad \frac{}{-\neg A} \\
\end{array}
\qquad
-\neg E: \quad \frac{-\neg A \quad -A}{\beta}$$

Pairs of premises  $\alpha$  and  $\alpha^*$  can be restricted to atomic formulas in all the rules in which they occur (see Appendix 5.3.1). Proofs carry over to the rules with  $\perp$ . Thus the system satisfies Rumfitt's requirement that only atomic formulas are stipulated to be co-ordinated, and the complex case can be proved by induction. Conclusions marked by  $\beta$ , incidentally, can also be restricted to atomics (see Appendix 5.3.2), although this is not something Rumfitt requires.

Let's have a final look at the importance of these restrictions. It is a fair question to ask why Rumfitt does not require Reductio to be restricted to atomics. Ferreira's paraphrase of Rumfitt's position indicates that it is not clear where the difference in treatment comes from: 'a necessary condition for a sentence to have a coherent bilateral sense is that the acts of asserting it and denying it should be co-ordinated [...] Since the sense of a molecular sentence must be *fully* determined by the introduction and elimination rules of its principal connective [...], the co-ordination principles for arbitrary sentences must not be postulated [as they are for the atomic sentences]. They should rather follow from the rules and the co-ordination at the atomic level.' (Ferreira (2008): 1057) Ferreira shows that Reductio cannot be restricted to atomic discharged formulas. It follows that Smiley cannot be restricted in that way either. In his response, Rumfitt writes that 'if a sentence has a determinate bilateral sense, then the grounds for asserting it must not intersect the grounds for denying it. But it is no requirement of determinacy of bilateral sense that any grounds that contradict the denial are *ipso facto* grounds for assertion'. His argument appears to be that Reductio 'is not required for determinacy of bilateral sense', hence need not be restricted to atomics, as only half of it is part of Humberstone's bilateral intuitionist logic. Reductio is 'the hallmark of classicism, not of bilateralism.' (The last three quotes are from (Rumfitt (2008): 1061)<sup>3</sup> The argument depends, however, on the claim that Humberstone's logic succeeds in specifying a determinate bilateral meaning for the intuitionist connectives, a claim that Rumfitt seems to contradict on the next page of the same article and also in his previous papers, quite apart from the consequences of the argument given here that it is systems like  $\mathfrak{B}$ , not  $\mathfrak{I}$ , hence not Humberstone's system either, that specify the bilateral senses of the connectives. This once more suggests that it is not ARGUMENT 1 that is of importance in answering Dummett's challenge, as it is fairly unclear what its status is supposed to be. But there is no need to go into any further detail. We can follow Rumfitt and require restrictions of (intuitionist versions of)

<sup>3</sup>It would not be entirely fair to say that the meaning of negation is not just given by the negation rules, but by them together with the co-ordination principles. The co-ordination principles codify the relation between assertion and denial. They are more fundamental than negation, which is defined in terms of them. So of course their relation has some influence on the behaviour of negation. The same is true for all other connectives. A question can be raised, though. Reductio is classical in spirit. Rumfitt claims that 'as a matter of simple definition, then, quite independently of the soundness of double negation elimination, the double conjugate  $\alpha^{**}$  is strictly identical with  $\alpha$  itself.' (Rumfitt (2000): 804) Consider the corresponding unilateral claim. We refrain from defining the meaning of negation by rules of inference and leave it primitive. We then use a negation rule corresponding to Smiley and claim that as a matter of simple definition,  $\neg\neg A$  is identical to  $A$ . This looks as if it prejudices issues between classicists and intuitionists.

Non-Contradiction but not Reductio, or alternatively the premises of Smiley but not its conclusion, without following his reasons for doing so.<sup>4</sup>  $\mathfrak{Int}\text{-}\mathfrak{B}$ , however, satisfies both requirements.

The rules of  $\mathfrak{Int}\text{-}\mathfrak{B}$  exhibit harmony in Dummett's sense: grounds and consequences of deriving rejected formulas balance each other. Reduction procedures for maximal formulas can be formulated in standard fashion and the system normalises. As shown in Appendix 5.3.3, they are also harmonious in Tennant's sense, approved by Rumfitt, where 'an introduction rule  $I$  is in *harmony* with an elimination rule  $E$  when (a)  $E$ 's major premiss expresses the weakest proposition that can be eliminated when using  $E$ , with  $I$  taken as given, and (b)  $I$ 's conclusion expresses the strongest proposition that can be introduced using  $I$ , with  $E$  taken as given.' (Rumfitt (2000): 790)  $\mathfrak{Int}\text{-}\mathfrak{B}$  shares all the meaning-theoretically relevant properties of  $\mathfrak{B}$ . Thus it endows the connectives with a coherent bilateral sense. This time, however, that sense is intuitionist. The claim that the negation rules of a bilateral intuitionist logic cannot be harmonious is incorrect. There is thus nothing specifically classicist about bilateralism.

To finish this section, consider what happens when the classical half of Smiley is added to  $\mathfrak{Int}\text{-}\mathfrak{B}$ :

Classical Smiley: From  $\Gamma, -A \vdash \alpha$  and  $\Gamma, -A \vdash \alpha^*$  infer  $\Gamma \vdash +A$

Then the rules of  $\mathfrak{UD}$  that are not part of  $\mathfrak{Int}\text{-}\mathfrak{B}$  are easily derivable. The classical half of Smiley can also be restricted to atomic premisses (see Appendix 5.3.4). Thus the result is a formalisation of a classical bilateral logic that, just like  $\mathfrak{B}$ , satisfies all of Rumfitt's meaning-theoretical requirements on a system intended to specify the bilateral senses of the connectives. Let's call this alternative system of classical bilateral logic  $\mathfrak{B}'$ . The classical nature of  $\mathfrak{B}'$  resides, not in the rules governing negation or another connective, but purely in this one structural rule.<sup>5</sup>

Some readers of Rumfitt may suspect that the classical nature of his bilateral system has been smuggled in via the structural rule Smiley or Reductio.  $\mathfrak{B}'$  gives this suspicion some more substance: the classical nature of the system is isolated and pinned down to the structural rule Classical Smiley. The question about which logic is the right one has thus been pushed from the operational rules governing the connectives, as was Dummett's proposal, to the structural rules of the system. Dummett gave criteria for singling out justified operational rules. Rumfitt has not provided a similar proposal to single out justified structural rules. Neither the criterion of harmony nor of the restriction of premisses to atomic formulas achieves this. If this gap can be filled, we have a new, bilateralist justification of classical logic. Alternatively, we may leave things there and accept that both, classical as well as intuitionist logic, can be bilaterally justified.<sup>6</sup>

<sup>4</sup>Ferreira's result can be established purely proof-theoretically, incidentally. A proof of  $+B \vee \neg B$  in normal form can only end with an application of Smiley or Reductio. The result follows once it is established that proofs in Rumfitt's system have a structure comparable to proofs in unilateral systems that normalise.

<sup>5</sup>I owe this observation, the neat way of putting it, and the suggestion to add this closing discussion, to a referee for this journal.

<sup>6</sup>See also footnote 3. In the context of Rumfitt's system with its classical negation rules, there is some justification for the claim that we can simply lay down by definition that  $\alpha$  and  $\alpha^*$  are identical. However, once we have an intuitionist bilateral system to consider, this won't do justice

We can also draw a positive conclusion. Comparing  $\mathfrak{Int}\text{-}\mathfrak{B}$  and  $\mathfrak{B}'$  allows us to say that classical and intuitionist logicians mean the same by negation, as negation is governed by the same operational rules in both systems. They only differ in the logical laws they think are valid for it, which is now a question of the structural rules. This is a good thing, as then the intuitionist need not accuse the classicist of making no sense with his negation unless we restrict propositions to decidable ones, nor need the classicist accuse the intuitionist of changing the topic and importing epistemic concerns into the meaning of negation. Such a position is hard to justify in a unilateral framework, and so bilateralism can claim this advantage over unilateralism. But the question remains how to decide whether or not to add Classical Smiley, and the conclusion that bilateralism goes with classical as well as with intuitionist logic stands.

## 4 Conclusion

I conclude that Rumfitt has not answered the strong reading of Dummett's challenge: it is not true that bilateralism excludes intuitionist logic as not being in a position to endow the connectives with a coherent bilateral meaning. It is possible to formalise harmonious rules for classical as well as intuitionist bilateral logics. As such, bilateralism is weaker than Dummett's unilateral position, which, Rumfitt agrees, excludes classical negation from having a coherent meaning that is determined by rules of inference.

Rumfitt can of course fall back on the weak Dummettian challenge and be content with having proposed a semantic theory on which classical logic is justified, even though it also allows intuitionist logic to be justified. However, in this case Rumfitt's position itself is also significantly weakened. If bilateralism can accommodate both logics, Rumfitt's new criterion for how to decide whether an area of discourse should be treated classically or intuitionistically, i.e. whether it is best treated bilaterally or unilaterally, must be abandoned. For instance, that the past is best treated bilaterally is no longer a reason for accepting classical logic for it, as we could just as well treat it intuitionistically and bilaterally. The stronger challenge is the philosophically more interesting one, and as Dummett himself establishes the stronger result on the basis of his semantic theory of excluding classical logic while justifying intuitionist logic, bilateralism loses much of its attraction.<sup>7</sup>

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to what is at issue in the debate.

<sup>7</sup>I would like to thank Dorothy Edgington, Giulia Felappi, Keith Hossack, Guy Longworth and Mark Textor for comments on previous versions of this paper. I also profited from presenting it in Shalom Lappin's Language and Cognition Seminar. A referee for this journal made valuable suggestions for clarifications and improvements.



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The proof uses only  $+ \neg E$ ,  $- \neg E$ ,  $+ \& E$ ,  $- \& E$ ,  $+ \vee E$ ,  $- \vee E$ ,  $+ \supset E$ ,  $- \supset E$ .

Case 2. Conjunction. Where  $C$  is an arbitrary atomic formula, the deduction on top can be transformed into the deduction below:

$$\frac{\frac{\overline{\alpha}^1}{\Pi_1} + A \& B}{\alpha^*} \quad \frac{\frac{\overline{\alpha}^1}{\Pi_2} - A \& B}{1}$$

Case 3. Disjunction. The deduction on top can be transformed into the deduction below, where  $C$  is an arbitrary atomic formula:



deduction the sum the degrees of its maximal formulas. Applying the reduction procedures to the left most and upper most maximal formula of a deduction will reduce its degree. Continuing in the same way will eventually produce a deduction *in normal form*, in which there are no maximal formulas.

Case 1. Conjunction. We need to consider only two cases. The obvious reduction procedure takes care of the case where  $+&I$  is followed immediately by  $+&E$ . The case where  $+&E$  is preceded by Reductio is dealt with by replacing the deduction on the left with the deduction on the right:

$$\begin{array}{c}
 \frac{}{-A \& B}^1 \\
 \Pi \\
 \frac{\perp}{+A \& B}^1 \\
 \frac{}{+A}
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \frac{}{+A \& B}^1 \\
 \frac{}{+A} \quad \frac{}{-A}^2 \\
 \frac{\perp}{-A \& B}^1 \\
 \Pi \\
 \frac{\perp}{+A}^2
 \end{array}$$

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If the conclusion  $+A$  is subject to an elimination rule, the reduction procedure may introduce a new maximal formula, but one that is of lower degree than the one removed. The case for the other elimination rule of  $+&$  is similar.

Case 2. Disjunction. We need to consider only two cases. The obvious reduction procedure takes care of the case where  $- \vee I$  is followed immediately by  $- \vee E$ . The case where  $- \vee E$  is preceded by Reductio is dealt with by replacing the deduction on the left with the deduction on the right:

$$\frac{\frac{\frac{}{+ A \vee B}^1 \quad \Pi}{\perp}^1}{- A \vee B}^1 \quad \rightsquigarrow \quad \frac{\frac{\frac{}{- A \vee B}^1}{- A} \quad \frac{}{+ A}^2}{+ A \vee B}^1 \quad \frac{\Pi}{\perp} \quad \frac{}{- A}^2$$

If the conclusion  $- A$  is subject to an elimination rule, the reduction procedure may introduce a new maximal formula, but one that is of lower degree than the one removed. The case for the other elimination rule of  $- \vee$  is similar.

Case 3. Negation. We need to consider only one case, which is dealt with by replacing the deduction on the left with the deduction on the right:

$$\frac{\frac{\frac{}{- \neg A}^1 \quad \Pi}{\perp}^1}{+ \neg A}^1 \quad \rightsquigarrow \quad \frac{\frac{\frac{}{+ \neg A}^1}{- A} \quad \frac{}{+ A}^2}{\perp}^1 \quad \frac{}{- \neg A}^1 \quad \frac{\Pi}{\perp} \quad \frac{}{- A}^2$$

If the conclusion  $- A$  is subject to an elimination rule, the reduction procedure may introduce a new maximal formula, but one that is of lower degree than the one removed.

Case 4. Implication. We need to consider three cases, corresponding to the three elimination rules for  $\supset$ :

i)

$$\frac{\frac{\frac{}{-A \supset B}^1}{\Pi} \quad \frac{\frac{}{+A \supset B}^1 \quad \frac{}{+A}}{\Sigma}}{\frac{}{+B}} \rightsquigarrow \frac{\frac{\frac{}{+A \supset B}^1 \quad \frac{}{+A}}{\Sigma} \quad \frac{\frac{}{-B}^2}{\frac{}{-A \supset B}^1}}{\Pi} \quad \frac{}{+B}^2$$

ii)

$$\frac{\frac{\frac{}{+A \supset B}^1}{\Pi} \quad \frac{}{+A}}{\frac{}{-A \supset B}^1} \rightsquigarrow \frac{\frac{\frac{}{-A \supset B}^1}{\frac{}{+A \supset B}^1} \quad \frac{}{-A}^2}{\Pi} \quad \frac{}{+A}^2$$

iii)

$$\frac{\frac{\frac{}{+A \supset B}^1}{\Pi} \quad \frac{}{-B}}{\frac{}{-A \supset B}^1} \rightsquigarrow \frac{\frac{\frac{}{-A \supset B}^1}{\frac{}{-B}} \quad \frac{}{+B}^2}{\Pi} \quad \frac{}{-B}^2$$

If the conclusions  $+A$ ,  $+B$  and  $-B$  are subject to an elimination rule, the reduction procedure may introduce a new maximal formula, but one that is of lower degree than the one removed.

Applying the reduction procedures for  $\supset$  may introduce maximal formulas of the same degree as the one removed at the top of the deduction  $\Pi$ , if  $+A \supset B$  was the premise of an elimination rule. For each possibility, however, the formula can be removed trivially, so that any maximal formula that might arise is then of lower complexity than the one removed.

In case i), if  $-A \supset B$  is major premise of an elimination rule, the conclusion is either  $+A$  or  $-B$ . In the former case, move  $\Sigma$  on top of  $+A$  and delete the rest. In the other case, delete everything above that conclusion and discharge it when applying Smiley to the conclusion  $\perp$  of  $\Pi$ .

If  $+A \supset B$  is major premise of an elimination rule, its conclusion is  $+B$  and there is a deduction  $\Sigma$  of the minor premise  $+A$ . In case ii), as the minor premise of the rule is identical to the conclusion of the whole deduction, move  $\Sigma$  on top of it and delete everything else. In case iii), delete everything on top of the conclusion  $+B$  of the rule and discharge it with the final application of Reductio.

### 5.3 Properties of $\mathfrak{Int}\text{-}\mathfrak{B}$

#### 5.3.1 Restrictions to Atomic Premises

- $\mathfrak{N}$
1. Intuitionist Smiley can be reduced to atomic premises  $\alpha$ . The proof appeals in the induction step to *ex contradictione quodlibet*,  $-\neg E$  and the first  $-\supset E$  restricted to atomic conclusions.
  2. *Ex contradictione quodlibet* can be restricted to atomic premises. The proof is obvious. It only needs the induction hypothesis in the steps for conjunctive and disjunctive premises. The step for  $+A \supset B, -A \supset B \vdash \beta$  appeals to Intuitionist Smiley, with premises  $+B, -B$  and conclusion  $-A$ , discharging  $+A$ .
  3. It follows immediately that  $\alpha$  can be restricted to atomics in  $-\supset I$  and  $-\neg I$ .

#### 5.3.2 Restrictions to Atomic Conclusions

1. *Ex falso quodlibet* can be restricted to atomic conclusions. The proof is obvious.
2. The same constructions show that the first  $-\supset I$  rule and  $-\neg E$  can be restricted to atomic conclusions.
3. Intuitionist Smiley can be restricted to atomic conclusions. The proof appeals to *ex contradictione quodlibet* in the step for  $\supset$ , to derive the premise  $+A \supset B$ .

### 5.3.3 The rules are harmonious in Tennant's sense

I'll only show this for the rejective implication and conjunction rules and leave negation to the reader.

1.a) Given  $\neg \supset I$ ,  $\neg A \supset B$  is the weakest speech act that can be eliminated by  $\neg \supset E$ . Suppose some speech act  $\phi$  satisfies the conditions specified for the major premise in the rules for  $\neg \supset E$ , i.e. for arbitrary  $C$ ,  $\phi, \neg A \vdash \neg C$ ,  $\phi, \neg A \vdash + C$  and  $\phi \vdash \neg B$ . Then by  $\neg \supset I$ ,  $\phi \vdash \neg A \supset B$ .

1.b) Given  $\neg \supset E$ ,  $\neg A \supset B$  is the strongest speech act that can be introduced by  $\neg \supset I$ . Suppose some speech act  $\phi$  satisfies the condition specified for the conclusion of the introduction rule for  $\neg \supset$ , i.e. if  $\Gamma, \neg A \vdash \neg C$ ,  $\Gamma, \neg A \vdash + C$  and  $\Gamma \vdash \neg B$ , then  $\Gamma \vdash \phi$ . By  $\neg \supset E$ ,  $\neg A \supset B, \neg A \vdash \neg C, \neg A \supset B, \neg A \vdash + C$  and  $\neg A \supset B \vdash \neg B$ . Hence by assumption  $\neg A \supset B \vdash \phi$ .

2.a) Given  $\neg \& I$ ,  $\neg A \& B$  is the weakest speech act that can be eliminated by  $\neg \& E$ . Suppose some speech act  $\phi$  satisfies the conditions specified for the major premise in  $\neg \& E$ , i.e.  $\phi, + A \vdash \neg B$ . Then by  $\neg \& I$ ,  $\phi \vdash \neg A \& B$ .

2.b) Given  $\neg \& E$ ,  $\neg A \& B$  is the strongest speech act that can be introduced by  $\neg \& I$ . Suppose some speech act  $\phi$  satisfies the condition specified for the conclusion of the introduction rule of  $\neg \&$ , i.e. if  $\Gamma, + A \vdash \neg B$ , then  $\Gamma \vdash \neg A \& B$ . By  $\neg \& E$ ,  $\neg A \& B, + A \vdash \neg B$ . Hence by assumption  $\neg A \& B \vdash \phi$ .



### 5.3.4 Adding the classical half of Smiley

The premises of the classical half of Smiley can be restricted to atomics when added to  $\mathfrak{Int}\text{-}\mathfrak{B}$ . We only need to consider the cases of  $\&$ ,  $\supset$  and  $\neg$ , the rules for  $\vee$  being as in  $\mathfrak{B}$ .

Case 1. Conjunction. The deduction on the left can be transformed into the deduction on the right:

$$\begin{array}{ccc}
 \frac{\frac{\overline{-C}^1}{\Pi_1} \quad \frac{\overline{-C}^1}{\Pi_2}}{+A\&B \quad -A\&B}_1 & \rightsquigarrow & \frac{\frac{\overline{-C}^1}{\Pi_2} \quad \frac{\overline{-C}^1}{\Pi_2} \quad \frac{\overline{-C}^1}{\Pi_2}}{+A\&B \quad -A\&B \quad +A\&B}_1 \\
 +C & & +C
 \end{array}$$

Case 2. Implication. Where  $D$  is an arbitrary atomic formula, the deduction on top can be transformed into the deduction below:

$$\begin{array}{ccc}
 \frac{\frac{\overline{-C}^1}{\Pi_1} \quad \frac{\overline{-C}^1}{\Pi_2}}{+A \supset B \quad -A \supset B}_1 & & \\
 +C & & \\
 \rightsquigarrow & & \\
 \frac{\frac{\frac{\overline{-C}^2}{\Pi_1} \quad \frac{\overline{-C}^2}{\Pi_2}}{+A \supset B \quad -A \supset B}_1 \quad \frac{\overline{-C}^2}{\Pi_2} \quad \frac{\overline{-C}^2}{\Pi_2} \quad \frac{\overline{-C}^2}{\Pi_2}}{+A \supset B \quad -A \supset B \quad -A \supset B}_1 & & \frac{\frac{\overline{-C}^2}{\Pi_1} \quad \frac{\overline{-C}^2}{\Pi_2} \quad \frac{\overline{-C}^2}{\Pi_2}}{+A \supset B \quad -A \supset B \quad -A \supset B}_1 \\
 +D & & -D}_2 \\
 +C & &
 \end{array}$$

Case 3. Negation. Where  $D$  is an atomic formula, the deduction on the left can be transformed into the deduction on the right:

$$\begin{array}{c}
\overline{-C}^1 \quad \overline{-C}^1 \\
\Pi_1 \quad \Pi_2 \\
\hline
+ \neg A \quad - \neg A \quad 1 \\
+ C
\end{array}
\rightsquigarrow
\begin{array}{c}
\overline{-C}^1 \quad \overline{-C}^1 \quad \overline{-C}^1 \quad \overline{-C}^1 \\
\Pi_1 \quad \Pi_2 \quad \Pi_1 \quad \Pi_2 \\
\hline
- \neg A \quad \frac{- \neg A}{- A} \quad - \neg A \quad \frac{- \neg A}{- A} \\
\hline
+ D \quad \quad \quad - D \quad 1 \\
+ C
\end{array}$$